PROJECTIONS IN DUALS TO ASPLUND SPACES MADE WITHOUT SIMONS' LEMMA

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In honor of the 70th birthday of Charles Stegall.

ABSTRACT. G. Godefroy and the second author of this note proved in 1988 that in duals to Asplund spaces there always exists a projectional resolution of the identity. A few years later, Ch. Stegall succeeded to drop from the original proof a deep lemma of S. Simons. Here, we rewrite the condensed argument of Ch. Stegall in a more transparent and detailed way. We actually show that this technology of Ch. Stegall leads to a bit stronger/richer object —the so called projectional skeleton— recently constructed by W. Kubiś, via S. Simons' lemma and with help of elementary submodels from logic.

In 1988, G. Godefroy and the second named author of this note constructed in [FG] a projectional resolution of the identity in duals to general Asplund spaces; see, e.g., [F, Definitions 1.0.1 and 6.1.5]. A few years later, Ch. Stegall, presented a simplified variant of this construction by avoiding from the proof the use of S. Simons' lemma [F, Lemma 8.1.3] or of any other substitute of it. He published it, in a rather condensed form in [St]. In this note, we perform his argument in full details. Thus we believe that Ch. Stegall's approach, so far overlooked by audience, will attract a broader attention. Ch. Stegall's technology led us even to constructing a stronger/richer object, which includes the so called 1-*projectional skeleton*, introduced and studied recently by W. Kubiś, see [K, pages 765, 766], [KKL, pages 369, 370]. W. Kubiś' construction was based on S. Simons' lemma and was done by using elementary submodels from logic.

Let $(X, \|\cdot\|)$ be any Banach space. If V is a subspace of X and $x^* \in X^*$, then $x^*{\restriction}_V$ means the restriction of x^* to V; similarly, we put $M{\restriction}_V := \{x^*{\restriction}_V \colon\ x^* \in M\}$ for any set M in X^*. Alaoglu's theorem asserts that the closed unit ball B_{X^*} in X^* provided with the weak* topology is a compact space. Let $\mathcal{C}(B_{X^*})$ denote the (Banach) space of all continuous functions on this compact space, endowed

with the maximum norm $\|\cdot\|$. Consider the multivalued mapping

$$\mathcal{C}(B_{X^*}) \ni f \longmapsto \{x^* \in B_{X^*} : \ f(x^*) = \max f(B_{X^*})\} =: \partial(f) \subset B_{X^*};$$

thus $\partial : \mathcal{C}(B_{X^*}) \longrightarrow 2^{X^*}$. Clearly, for every $f \in \mathcal{C}(B_{X^*})$ the set $\partial(f)$ is non-empty and weak* compact. The mapping ∂ is also norm-to-weak* upper semi-continuous. To check this, consider any weak* open set W in X^*. We have to show that $\{f \in \mathcal{C}(B_{X^*}) : \ \partial(f) \subset W\}$ is an open set. Assume that this is not the case. Then there exists a sequence $f_0, f_1, f_2, \ldots$ of elements in $\mathcal{C}(B_{X^*})$ such that $\partial(f_0) \subset W$, that $\|f_n - f_0\| \to 0$ as $n \to \infty$, and that for every $n \in \mathbb{N}$ there is $x_n^* \in \partial(f_n) \setminus W$. Recalling that all the x_n^*'s belong to the (weak* compact) set B_{X^*}, the sequence (x_n^*) has a weak* cluster point, $x^* \in B_{X^*}$, say. Also

$$\begin{aligned}
\max f_0(B_{X^*}) \ &\leq \ \max f_n(B_{X^*}) + \|f_0 - f_n\| \\
&= \ f_n(x_n^*) - f_0(x_n^*) + \|f_0 - f_n\| + f_0(x_n^*) \leq 2\|f_n - f_0\| + f_0(x_n^*)
\end{aligned}$$

for every $n \in \mathbb{N}$. And, since f_0 is weak* continuous, $\big(f_0(x^*) \leq\big)$ $\max f_0(B_{X^*}) \leq f_0(x^*)$. We got that $x^* \in \partial(f_0)$ $(\subset W)$. However, simultaneously, $x^* \notin W$ as $x_n^* \notin W$ for all $n \in \mathbb{N}$; a contradiction.

If the Banach space $(X, \|\cdot\|)$ is separable, the mapping $\partial : \mathcal{C}(B_{X^*}) \longrightarrow 2^{X^*}$ has the following (crucial) property: For every $\xi \in B_{X^*}$ there is an $f \in \mathcal{C}(B_{X^*})$ such that $\partial(f) = \{\xi\}$. (This is the very point where the approach of Ch. Stegall starts.) Indeed, since the compact space (B_{X^*}, w^*) is then metrizable, ξ is a G_δ point in it and Urysohn's lemma [E, Theorem 1.5.10] provides easily a suitable function f. More specifically (and without using any topological tool), such an f can be constructed by the formula

$$(1) \qquad B_{X^*} \ni x^* \longmapsto 1 - \sum_{n=1}^{\infty} 2^{-n}\big|\langle \xi, s_n\rangle - \langle x^*, s_n\rangle\big| =: f(x^*)$$

where $\{s_1, s_2, \ldots\}$ is a fixed countable dense subset of the closed unit ball B_X in X. (By [F, Lemma 2.2.1(ii)] we know that the function $\mathcal{C}(B_{X^*}) \ni g \longmapsto \max g(B_{X^*})$ is Gateaux differentiable at $g := f$.)

In what follows, let $(X, \|\cdot\|)$ be a Banach space of an arbitrary density. The observation in the latter paragraph leads to introducing the following family of functions from $\mathcal{C}(B_{X^*})$. Let S be any nonempty set in B_X. By $\mathcal{L}(S)$ we denote the family of all functions of the form

$$B_{X^*} \ni x^* \longmapsto 1 - \sum_{n=1}^{k} 2^{-n}\big|a_n - \langle x^*, s_n\rangle\big|,$$

where $k \in \mathbb{N}$, $a_1, a_2, \ldots, a_k$ are rational numbers in $[-1, 1]$, and $s_1, s_2, \ldots, s_k$ are elements from S; note that $\#\mathcal{L}(S) = \#S + \aleph_0$. It is easy to check that each element of $\mathcal{L}(S)$ is weak* continuous and has maximum norm at most equal to 1. Thus $\mathcal{L}(S)$ is a subset of (the closed unit ball of) $\mathcal{C}(B_{X^*})$. We can easily

check that

(2) $$\overline{\mathcal{L}(\overline{S})} = \overline{\mathcal{L}(S)};$$

hence, the set in (2) is separable whenever S is separable. Note that the f defined in (1) belongs to the (norm) closure of $\mathcal{L}(\{s_1, s_2, \ldots\})$.

From now on, assume that $(X, \|\cdot\|)$ is an Asplund space, which (equivalently) means that the dual space X^* is weak* dentable [Ph, Theorem 2.32]. Then, we are ready to apply the selection theorem of Jayne and Rogers [JR, Theorem 8], [F, Theorem 8.1.2] to our multivalued mapping ∂ (which is already known to be norm-to-weak* upper semicontinuous and weak* compact valued). Thus we get a sequence $\lambda_j : \mathcal{C}(B_{X^*}) \longrightarrow X^*$, $j \in \mathbb{N}$, of norm-to-norm continuous mappings such that for every $f \in \mathcal{C}(B_{X^*})$ the limit $\lim_{j \to \infty} \lambda_j(f) =: \lambda_0(f)$ exists in the norm topology of X^* and moreover $\lambda_0(f) \in \partial(f)$, that is, $f(\lambda_0(f)) = \max f(B_{X^*})$. Now, we define the multivalued mapping

$$\mathcal{C}(B_{X^*}) \ni f \longmapsto \{\lambda_1(f), \lambda_2(f), \ldots\} =: \Lambda(f) \subset X^*;$$

thus $\Lambda : \mathcal{C}(B_{X^*}) \longrightarrow 2^{X^*}$. The continuity of the mappings λ_j's and (2) then guarantee that

(3) $$\overline{\Lambda(\overline{\mathcal{L}(\overline{S})})} = \overline{\Lambda(\mathcal{L}(S))}.$$

Proposition 1. *Let $(X, \|\cdot\|)$ be any Asplund space and let V be any subspace of it. Then $B_{V^*} \subset \overline{\Lambda(\mathcal{L}(B_V))} \restriction_V$ and $\operatorname{dens} V = \operatorname{dens} \operatorname{sp} \overline{\Lambda(\mathcal{L}(B_V))} = \operatorname{dens} V^*$; in particular, $B_{X^*} \subset \overline{\Lambda(\mathcal{L}(B_X))}$.*

Proof. First, assume that V is separable. Consider any $v^* \in B_{V^*}$. Find a countable dense subset $\{s_1, s_2, \ldots\}$ of B_V. Define $f := 1 - \sum_{n=1}^{\infty} 2^{-n} |\langle v^*, s_n \rangle - \langle \cdot, s_n \rangle|$; thus $f \in \overline{\mathcal{L}(B_V)}$. Then $f(x^*) = 1$ for every $x^* \in B_{X^*}$, with $x^* \restriction_V = v^*$, and $-1 \leq f(y^*) < 1$ whenever $y^* \in B_{X^*}$ and $y^* \restriction_V \neq v^*$. Thus $\partial(f) \restriction_V = \{v^*\}$, and, since $\lambda_0(f) \in \partial(f)$, we have that $\lambda_0(f) \restriction_V = v^*$. Recalling that $\|\lambda_j(f) - \lambda_0(f)\| \longrightarrow 0$ as $j \to \infty$, we can conclude that

$$\lambda_0(f) \in \overline{\Lambda(f)} \subset \overline{\Lambda(\mathcal{L}(B_V))} = \overline{\Lambda(\mathcal{L}(B_V))}$$

by (3). Therefore, $v^* = \lambda_0(f) \restriction_V \in \overline{\Lambda(\mathcal{L}(B_V))} \restriction_V$.

Second, assume that V is non-separable (provided that X is non-separable). Consider any $v^* \in B_{V^*}$. Find $\xi \in B_{X^*}$ so that $\xi \restriction_V = v^*$. We shall construct countable sets $S_0 \subset S_1 \subset S_2 \subset \cdots \subset B_V$ and separable subspaces $Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset V$ as follows. Pick any countable subset S_0 of B_V and any separable (rather infinite-dimensional) subspace $Z_0 \subset V$. Let $m \in \mathbb{N}$ be given and assume that we have found S_{m-1} and Z_{m-1}. Clearly, $\Lambda(\mathcal{L}(S_{m-1}))$ is a countable subset of X^*. Find then a countable set $S_{m-1} \subset S_m \subset B_V$ such that $\overline{S_m} \supset B_X \cap Z_{m-1}$

and that

(4) $\quad \|\xi - x^*\| = \sup\left\{\langle\xi, s\rangle - \langle x^*, s\rangle : \ s \in S_m\right\} \ \text{ for every } x^* \in \Lambda\big(\mathcal{L}(S_{m-1})\big).$

Put then $Z_m := \overline{\text{sp}\,(Z_{m-1} \cup S_m)}.$ Do so for every $m \in \mathbb{N}$ and put finally $S := S_0 \cup S_1 \cup S_2 \cup \cdots$ and $Z := \overline{Z_0 \cup Z_1 \cup Z_2 \cup \cdots}.$ Clearly, S is a countable set, Z is a separable subspace of V, and $\overline{S} = B_Z.$ The "separable" case says that $\xi{\upharpoonright}_Z \in \overline{\Lambda\big(\mathcal{L}(B_Z)\big)}{\upharpoonright}_Z.$ By (3) we get that $\xi{\upharpoonright}_Z$ actually belongs to the set $\overline{\Lambda(\mathcal{L}(S))}{\upharpoonright}_Z.$ Pick $x^* \in \overline{\Lambda(\mathcal{L}(S))}$ so that $\xi{\upharpoonright}_Z = x^*{\upharpoonright}_Z.$ For every $i \in \mathbb{N}$ find $x_i^* \in \Lambda(\mathcal{L}(S))$ so that $\|x^* - x_i^*\| < \frac{1}{i}.$ A moment reflection reveals that for every $i \in \mathbb{N}$ there is $m_i \in \mathbb{N}$ so that $x_i^* \in \Lambda(\mathcal{L}(S_{m_i-1})).$ Then, by (4),

$$\begin{aligned}
\|\xi - x_i^*\| &= \sup\left\{\langle\xi, s\rangle - \langle x_i^*, s\rangle : \ s \in S_{m_i}\right\} \\
&\leq \ \big\|\xi{\upharpoonright}_Z - x_i^*{\upharpoonright}_Z\big\| = \big\|x^*{\upharpoonright}_Z - x_i^*{\upharpoonright}_Z\big\| \leq \|x^* - x_i^*\| < \tfrac{1}{i}
\end{aligned}$$

for every $i \in \mathbb{N}.$ Therefore $\xi \in \overline{\Lambda(\mathcal{L}(S))} \subset \overline{\Lambda(\mathcal{L}(B_V))},$ and so, $v^* = \xi{\upharpoonright}_V \in \overline{\Lambda\big(\mathcal{L}(B_V)\big)}{\upharpoonright}_V.$

Finally, consider any subspace V of $X.$ Choosing a dense subset M of $B_V,$ with $\#M = \text{dens}\,V,$ we have by (3)

$$\begin{aligned}
\text{dens}\,V \ &\leq \ \text{dens}\,V^* \leq \text{dens}\,\overline{\Lambda\big(\mathcal{L}(B_V)\big)}{\upharpoonright}_V \leq \ \text{dens}\,\overline{\Lambda\big(\mathcal{L}(B_V)\big)} = \text{dens}\,\overline{\text{sp}\Lambda\big(\mathcal{L}(B_V)\big)} \\
&= \ \text{dens}\,\overline{\Lambda(\mathcal{L}(M))} \leq \#\Lambda\big(\mathcal{L}(M)\big) = \#M \ = \text{dens}\,V.
\end{aligned}$$

$\square$

Remark 2. Proposition 1 is crucial for dropping S. Simons' lemma from the original construction of a projectional resolution of the identity in duals to Asplund spaces; see [FG] modulo [F1]. Indeed, in [FG], instead of the mapping $\partial : \mathcal{C}(B_{X^*}) \longrightarrow 2^{B_{X^*}},$ there is considered the (so called duality) mapping $X \ni x \longmapsto \{x^* \in S_{X^*} : \ \langle x^*, x\rangle = \|x\|\} =: J(x).$ This $J : X \longrightarrow 2^{S_{X^*}}$ is also norm-to-weak* upper semicontinuous and weak* compact valued. Hence, the Jayne-Rogers theorem [F, Theorem 8.1.2] yields norm-to-norm continuous mappings $D_1, D_2, \ldots$ (now) from X into X^* such that for every $x \in X$ the limit $\lim_{n\to\infty} D_n(x) =: D_0(x)$ exists in the norm topology of X^* and moreover $\langle D_0(x), x\rangle = \|x\|.$ Yet, there is no obvious guarantee that the set $D_0(X)$ is equal to, or at least norm-dense in the unit sphere $S_{X^*}.$ Here, S. Simons' lemma enters the argument and remedies the situation by showing that $D_0(X)$ is linearly dense in all of $X^*.$

Proposition 3. *Let $(X, \|\cdot\|)$ be a non-separable Asplund space and let $Z \subset X$ be an infinite-dimensional subspace with $\text{dens}\,Z < \text{dens}\,X.$ Then there exists an overspace $Z \subset V \subset X,$ with $\text{dens}\,V = \text{dens}\,Z,$ such that the restriction mapping $\text{sp}\,\Lambda\big(\mathcal{L}(B_V)\big) \ni x^* \longmapsto x^*{\upharpoonright}_V =: R(x^*) \in V^*$ is a (surjective) isometry.*

Proof. Put $\aleph := \text{dens}\,Z.$ By induction, we shall construct sets $S_0 \subset S_1 \subset S_2 \subset \cdots \subset B_X,$ all of cardinality $\aleph,$ as follows. Let S_0 be a dense subset of $B_Z,$ with

$\#S_0 = \aleph$. Let $m \in \mathbb{N}$ be fixed and assume that we have already found S_{m-1}. We already know that $\#\mathcal{L}(S_{m-1}) = \aleph$. Hence, by the norm-to-norm continuity of the mappings λ_j's, we have that $\overline{\mathrm{sp}\,\Lambda\big(\mathcal{L}(S_{m-1})\big)}$ is a subspace of X^*, with density $\aleph$; let M be a dense set in it, with $\#M = \aleph$. Find a set $S_{m-1} \subset S_m \subset B_X$, with $\#S_m = \aleph$, so big that $\mathrm{sp}\,S_{m-1} \cap B_X \subset \overline{S_m}$ and that $\|\xi\| = \sup\langle \xi, S_m\rangle$ for every $\xi \in M$. Then, of course, $\|x^*\| = \sup\langle x^*, S_m\rangle$ for every $x^* \in \overline{\mathrm{sp}\,\Lambda\big(\mathcal{L}(S_{m-1})\big)}$. This finishes the induction (step).

Having constructed the S_m for every $m \in \mathbb{N}$, put $S := S_1 \cup S_2 \cup \cdots$ and $V := \overline{\mathrm{sp}\,S}$; then $\#S = \aleph$ and V is a subspace of X, with density $\aleph$. We shall show that this V, together with the corresponding R, serve for the conclusion of our proposition. Consider any $x^* \in \overline{\mathrm{sp}\,\Lambda\big(\mathcal{L}(B_V)\big)}$ and let $\varepsilon > 0$ be arbitrary. It is easy to verify that $\overline{S} = B_V$. By (3), $\overline{\Lambda(\mathcal{L}(B_V))} = \overline{\Lambda(\mathcal{L}(S))}$, and hence $\overline{\mathrm{sp}\,\Lambda(\mathcal{L}(B_V))} = \overline{\mathrm{sp}\,\Lambda(\mathcal{L}(S))}$. Thus x^* belongs to $\overline{\mathrm{sp}\,\Lambda\big(\mathcal{L}(S)\big)}$. Find then $\xi \in \mathrm{sp}\,\Lambda\big(\mathcal{L}(S)\big)$ so that $\|x^* - \xi\| < \varepsilon$. We remark that ξ belongs even to $\mathrm{sp}\,\Lambda\big(\mathcal{L}(S_{m-1})\big)$ for some big $m \in \mathbb{N}$. Then

$$\begin{aligned}
\|x^*\| - \varepsilon \;&<\; \|\xi\| = \sup\langle \xi, S_m\rangle \le \sup\langle \xi, B_V\rangle \\
&=\; \|R(\xi)\| < \|R(x^*)\| + \varepsilon \le \|x^*\| + \varepsilon.
\end{aligned}$$

Thus $\|x^*\| = \|R(x^*)\|$. We proved that R is an isometry. That R is surjective follows immediately from Proposition 1. $\qquad\square$

Proposition 4. *Let V be a subspace of an Asplund space $(X, \|\cdot\|)$ such that the restriction mapping $\overline{\mathrm{sp}\,\Lambda\big(\mathcal{L}(B_V)\big)} \ni x^* \longmapsto x^*\!\restriction_V =: R(x^*) \in V^*$ is a (surjective) isometry. Then the mapping $X^* \ni x^* \longmapsto R^{-1}\big(x^*\restriction_V\big) =: P(x^*)$ is a linear norm-1 projection, $P(X^*) = \overline{\mathrm{sp}\,\Lambda\big(\mathcal{L}(B_V)\big)}$, $\mathrm{dens}\,P(X^*) = \mathrm{dens}\,V$, and $\overline{V}^{\,w^*} = P^*(X^{**})$.*

Proof. The first three statements concerning P immediately follow from the definition of it. The "density" statement is contained in Proposition 1. It remains to prove the last equality. That $V \subset P^*(X^{**})$ follows from the definition of P; hence $\overline{V}^{\,w^*} \subset P^*(X^{**})$. Assume there exists $x^{**} \in P^*(X^{**}) \setminus \overline{V}^{\,w^*}$. The Hahn-Banach separation theorem yields an $x^* \in X^*$ such that $\langle x^{**}, x^*\rangle \neq 0$ and $x^*\!\restriction_V \equiv 0$. But

$$\langle x^{**}, x^*\rangle = \langle P^*(x^{**}), x^*\rangle = \langle x^{**}, P(x^*)\rangle = \big\langle x^{**}, R^{-1}\big(x^*\restriction_V\big)\big\rangle = 0;$$

a contradiction. $\qquad\square$

Proposition 5. *Let V_1, V_2 be two subspaces of an Asplund space $(X, \|\cdot\|)$ such that $V_1 \subset V_2$ and that the restriction mappings $\overline{\mathrm{sp}\,\Lambda\big(\mathcal{L}(B_{V_i})\big)} \ni x^* \longmapsto x^*\!\restriction_{V_i} =: R_i(x^*) \in V_i^*$, $i = 1, 2$, are (surjective) isometries. Define $P_i : X^* \to X^*$ by $P_i(x^*) = R_i^{-1}\big(x^*\restriction_{V_i}\big)$, $x^* \in X^*$, $i = 1, 2$. Then $P_1 \circ P_2 = P_1$ $(= P_2 \circ P_1)$, and $(P_2 - P_1)(X^*)$ is isometrical with $(V_2/V_1)^*$.*

Proof. (i) From the definition of P_i's we have immediately that $P_2 \circ P_1 = P_1$. Now, consider any $x^* \in X^*$ and any $x^{**} \in X^{**}$. Since $\overline{V_1}^{w^*} = P_1^*(X^{**})$ by Proposition 4, there is a net $(v_\tau)_{\tau \in T}$ in V_1 which weak* converges to $P_1^*(x^{**})$. Then, using the definition of R_2 and the inclusion $V_1 \subset V_2$, we get

$$
\begin{aligned}
\langle P_2^* \circ P_1^*(x^{**}), x^* \rangle &= \langle P_1^*(x^{**}), P_2(x^*) \rangle = \lim_{\tau \in T} \langle P_2(x^*), v_\tau \rangle \\
&= \lim_{\tau \in T} \langle R_2^{-1}(x^* {\restriction}_{V_2}), v_\tau \rangle = \lim_{\tau \in T} \langle x^*, v_\tau \rangle = \langle P_1^*(x^{**}), x^* \rangle,
\end{aligned}
$$

and so $P_2^* \circ P_1^* = P_1^*$, that is, $P_1 \circ P_2 = P_1$. The "isometrical" statement can be shown as in the proof of [F, Proposition 6.1.9(iv)]. □

Now we are armed to construct a projectional resolution of the identity on the dual to every Asplund space. But we rather prefer to present a bit stronger/richer statement.

Theorem 6. *Let $(X, \| \cdot \|)$ be a non-separable Asplund space. Then there exist a family $\mathcal{V}$ of subspaces of X and a family $\{Y_V : V \in \mathcal{V}\}$ of subspaces of X^* such that*

(i) *$\bigcup \{V : V \in \mathcal{V} \text{ and } \dens V = \aleph\} = X$ and $\bigcup \{Y_V : V \in \mathcal{V} \text{ and } \dens V = \aleph\} = X^*$ for every infinite cardinal $\aleph < \dens X$;*

(ii) *if $V_1, V_2 \in \mathcal{V}$, there is $V \in \mathcal{V}$ such that $V \supset V_1 \cup V_2$ and $\dens V = \max\{\dens V_1, \dens V_2\}$.*

(iii) *for every $V \in \mathcal{V}$ the assignment $Y_V \ni x^* \longmapsto x^* {\restriction}_V =: R_V(x^*) \in V^*$ is a surjective isometry, and hence the mapping $X^* \ni x^* \longmapsto R_V^{-1}(x^* {\restriction}_V) =: P_V(x^*)$ is a norm-1 linear projection on X^*, with range Y_V, and $\dens P_V(X^*) = \dens V$;*

(iv) *$\overline{V}^{w^*} = P_V^*(X^{**})$ for every $V \in \mathcal{V}$;*

(v) *$\mathcal{V}$ is complete in the following sense: if γ is a limit ordinal, and $\{V_\alpha : 1 \le \alpha < \gamma\}$ is an increasing long sequence of elements of $\mathcal{V}$, then $V := \overline{\bigcup_{1 \le \alpha < \gamma} V_\alpha}$ belongs to $\mathcal{V}$ and $Y_V = \overline{\bigcup_{1 \le \alpha < \gamma} Y_{V_\alpha}}$;*

(vi) *if $V, U \in \mathcal{V}$ and $V \subset U$, then $Y_V \subset Y_U$, $P_V \circ P_U = P_V$ $(= P_U \circ P_V)$, and $(P_U - P_V)(X^*)$ is isometrical with $(U/V)^*$.*

Proof. For every subspace V of X we put $Y_V := \overline{\operatorname{sp} \Lambda(\mathcal{L}(B_V))}$ and we consider the assignment $Y_V \ni x^* \longmapsto x^* {\restriction}_V =: R_V(x^*) \in V^*$. Let $\mathcal{V}$ consist of all subspaces V of X such that $R_V : Y_V \longrightarrow V^*$ is a surjective isometry.

(i) and (ii) are guaranteed by Propositions 3 and 1.

(iii) follows from the definition of $\mathcal{V}$ via Propositions 1 and 4.

(iv) follows from Proposition 4.

(v) Assume the premise here holds. (3) and some elementary reasoning yields

$$\Lambda(\mathcal{L}(B_V)) \subset \overline{\Lambda(\mathcal{L}(B_V))} = \overline{\Lambda\big(\mathcal{L}(\overline{\bigcup_{\alpha<\gamma}B_{V_\alpha}})\big)} = \overline{\Lambda\big(\mathcal{L}(\bigcup_{\alpha<\gamma}B_{V_\alpha})\big)}$$

$$= \overline{\bigcup_{\alpha<\gamma}\Lambda\big(\mathcal{L}(B_{V_\alpha})\big)} \subset \overline{\bigcup_{\alpha<\gamma}\mathrm{sp}\Lambda\big(\mathcal{L}(B_{V_\alpha})\big)} = \overline{\bigcup_{\alpha<\gamma}Y_{V_\alpha}}.$$

Hence

(5) $$Y_V = \overline{\mathrm{sp}\Lambda\big(\mathcal{L}(B_V)\big)} = \overline{\bigcup_{\alpha<\gamma}Y_{V_\alpha}}.$$

It remains to show that $V \in \mathcal{V}$, that is, that the mapping $R_V : Y_V \longrightarrow V^*$ is a surjective isometry. By Proposition 1 and (5), R_V is surjective. Further, fix any $x^* \in Y_V$ and let $\varepsilon > 0$ be arbitrary. Find $\xi \in \bigcup_{\alpha<\gamma}Y_\alpha$ such that $\|x^* - \xi\| < \varepsilon$. Then ξ belongs to Y_α for some $\alpha < \gamma$. Now, as $V_\alpha \in \mathcal{V}$, we have

$$\|x^*\| - \varepsilon < \|\xi\| = \|R_{V_\alpha}(\xi)\| = \big\|\xi\restriction_{V_\alpha}\big\| \le \big\|\xi\restriction_V\big\|$$

$$< \big\|x^*\restriction_V\big\| + \varepsilon = \|R_V(x^*)\| + \varepsilon \le \|x^*\| + \varepsilon.$$

Therefore, $\|x^*\| = \|R_V(x^*)\|$ and the mapping R_V is shown to be an isometry. Thus $V \in \mathcal{V}$.

(vi) immediately follows from Proposition 5.

$\qquad\square$

Corollary 7. *Let* $(X, \|\cdot\|)$ *be a non-separable Asplund space. Then* $(X^*, \|\cdot\|)$ *admits*

(i) [FG] a projectional resolution of the identity, and also

(ii) [K] a 1-projectional skeleton.

Proof. (i) Let μ be the first ordinal with $\#\mu = \mathrm{dens}\,X$. Find a dense subset $\{x_\alpha : \omega < \alpha < \mu, \alpha \text{ is non-limit}\}$ in X. By (i) find a separable element $V_\omega \in \mathcal{V}$. Let $\gamma \in (\omega, \mu]$ be any ordinal an assume that we already found $V_\alpha \in \mathcal{V}$ for every $\omega \le \alpha < \gamma$. Assume first that γ is non-limit. By (i) find a $V \in \mathcal{V}$ such that $V \ni x_\gamma$ and $\mathrm{dens}\,V = \mathrm{dens}\,V_{\gamma-1}$. By (ii) find a $V_\gamma \in \mathcal{V}$ such that $V_\gamma \supset V \cup V_{\gamma-1}$ and $\mathrm{dens}\,V_\gamma = V_{\gamma-1}$. Second, assume that γ is a limit ordinal. Put then $V_\gamma = \overline{\bigcup_{\omega<\alpha<\gamma}V_\alpha}$. By (v), we have that $V_\gamma \in \mathcal{V}$ and $Y_{V_\gamma} = \overline{\bigcup_{\omega\le\alpha<\gamma}Y_{V_\alpha}}$. Now we can immediately verify that $\{P_{V_\alpha} : \omega \le \alpha \le \mu\}$ is a projectional resolution of the identity on $(X^*, \|\cdot\|)$; see, e.g. [F, Definition 6.1.5].

(ii) We note that the relation "$\subset$" on the family $\mathcal{V}$ from Theorem 6 is a directed partial order, which is moreover σ-complete. Thus, the subfamily $\{P_V : V \in \mathcal{V} \text{ and } \mathrm{dens}\,V = \aleph_0\}$ is a 1-projectional skeleton on $(X^*, \|\cdot\|)$ in the sense of the definition in [KKL, pages 369, 370]. $\qquad\square$

Remark 8. 1. A 1-projectional skeleton in the dual to an Asplund space was originally constructed by W. Kubiś [K]. He started from the existence of the so called *projectional generator* in the dual to an Asplund space [F, Proposition 8.2.1] and then he proceeded using elementary submodels from logic. It should

be noted that the proof of [F, Proposition 8.2.1] is based on S. Simons' lemma and on the Jayne-Rogers selection theorem [JR, Theorem 8].

2. Let $\| \cdot \|_1$, $\| \cdot \|_2$, ... be a sequence of equivalent norms on an Asplund space X. Let $\mathcal{V}_1$, $\mathcal{V}_2$, ... be corresponding families found in Theorem 6 for these norms. It is not difficult to show that the family $\mathcal{V} := \bigcap_{i=1}^{\infty} \mathcal{V}_i$ (is not only non-empty but that it even) satisfies all the conditions (i) – (vi) of Theorem 6; see the proof of [BM, Proposition 1.1]. Then Corollary 7 provides one projectional resolution of the identity and one 1-projectional skeleton on X^* which can be related to any of the norms $\| \cdot \|_1$, $\| \cdot \|_2$, ...

3. Theorem 6 can be proved also with help of tools from [FG] (where S. Simons' lemma was used). Indeed, let the mappings $D_n : X \to X^*$, $n \in \mathbb{N}$, be from Remark 2 (they come from the Jayne-Rogers theorem) and define $D : X \to 2^{X^*}$ by $D(x) = \{D_1(x), D_2(x), \ldots\}$, $x \in X$. For every subspace W of X put $Y_W := \overline{\mathrm{sp}\, D(W)}$ and consider the assignment $Y_W \ni x^* \longmapsto x^*{\upharpoonright}_W =: R_W(x^*) \in W^*$. Let $\mathcal{W}$ consist of all subspaces W of X such that $R_W : Y_W \longrightarrow W^*$ is a surjective isometry. From [FG, pages 145, 146], where S. Simons' lemma and other things were used, we know that

(6) $$\overline{\mathrm{sp}\, D(W){\upharpoonright}_W} = W^* \quad \text{for every subspace} \quad W \subset X.$$

(This is a weakened analogue of our Proposition 1). By [F1, Lemma 1], for every $Z \subset X$, with $\mathrm{dens}\, Z < \mathrm{dens}\, X$ there is a subspace $Z \subset W \subset X$, with $\mathrm{dens}\, W = \mathrm{dens}\, Z$, such that R_W is an isometry from Y_W into W^*. (This is an analogue of our Proposition 3.) Using this, we easily get that $\overline{\mathrm{sp}\, D(W){\upharpoonright}_W} = \overline{\mathrm{sp}\, D(W)}{\upharpoonright}_W$, and by (6), $R_W : Y_W \to W^*$ is a surjective isometry. Thus $W \in \mathcal{W}$. This way we get the properties (i) – (iv) and (vi) listed in Theorem 6. (v) can be proved as in Theorem 6, now from the norm-to-norm continuity of the D_n's.

4. We use an opportunity to fix some inaccuracy in the book [F]. In the proof of [F, Proposition 8.2.1], on the page 152, the equality (2) should read as $\overline{\mathrm{sp}\, \Phi(B_0){\upharpoonright}_Y} = Y^*$ and the set Δ should be defined as $\overline{\Phi(B_0){\upharpoonright}_Y} \cap B_{Y^*}$.

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